Canonical Nonabelian Dual Transformations in Supersymmetric Field Theories

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A generating functional F is found for a canonical nonabelian dual transformation which maps the supersymmetric chiral O(4) σ-model to an equivalent supersymmetric extension of the dual σ-model. This F produces a mapping between the classical phase spaces of the two theories in which the bosonic (coordinate) fields transform nonlocally, the fermions undergo a local tangent space chiral rotation, and all currents (fermionic and bosonic) mix locally. Purely bosonic curvature-free currents of the chiral model become a *symphysis* of purely bosonic and fermion bilinear currents of the dual theory. The corresponding transformation functional T which relates wavefunctions in the two quantum theories is argued to be *exactly* given by $T = \exp(iF)$.

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In a general sense, a dual transformation is a map between two models which preserves dynamical features and thereby facilitates our understanding by providing alternate "dual" views of these features. Examples include Kramers-Wannier duality, relating high and low temperature limits of lattice models; bosonization of fermions, to reveal properties of solitons [1]; alternate Lorentz tensor descriptions of spinning fields; mirror symmetries relating compactified spacetimes in string theories [2]; and recent non-perturbative methods in supersymmetric four dimensional field theories [3].

Here, we generate nonabelian field theory examples of dual transformations for nonlinear σ-models in two spacetime dimensions, within the framework of equal-time canonical transformations, in parallel to earlier abelian work for the quantum Liouville theory [4]. We have already illustrated this equal-time mapping approach for purely bosonic σ-models [5]. We first summarize and clarify those earlier results by comparing to the lagrangean framework. We then construct new mappings by including fermions supersymmetrically. Our discussion concentrates on classical field theory, but we believe an acceptable route to quantization exists for these σ-models exactly as it does for the Liouville case: by simply exponentiating the classical generating functional, which we discuss in concluding.

For the supersymmetric case, our new dual transformation does not separately map bosons to bosons and fermions to fermions; rather it ties together the effects of both fermions and bosons in one model to produce the effects of either bosons or fermions separately in the other, dual model. This occurs even though the map does not interchange fermions with bosons (e.g. as in the soliton studies mentioned). While this phenomenon might be prefigured in the current identifications of canonical bosonization schemes [1], we believe its novelty warrants introduction of the term "boson-fermion symphysis".

To begin, recall the standard bosonic Chiral Model (CM) on O(4) $\simeq$ O(3)$\times$O(3)$\simeq$SU(2)$\times$SU(2). Expressed in geometrical terms, we have $\mathcal{L}_{CM} = \frac{1}{2} g_{ab} \partial_\mu \varphi^a \partial^\mu \varphi^b$, where g_{ab} is the metric on the field manifold (three-sphere). Explicitly, with group elements parameterized as $U = \varphi^0 + i\tau^j \varphi^j$, $(j = 1, 2, 3)$, where $(\varphi^0)^2 + \varphi^2 = 1$, and $\varphi^2 \equiv \sum_j (\varphi^j)^2$, we may resolve $\varphi^0 = \pm\sqrt{1-\varphi^2}$, to obtain $g_{ab} = \delta^{ab} + \varphi^a \varphi^b/(1-\varphi^2)$. We re-express the action in terms of tangent quantities through the use of either left- or right-invariant vielbeine (e.g. [6]). Either choice yields the Sugawara current-current form $\mathcal{L}_{CM} = \frac{1}{2} J^j_\mu J^{\mu\ j}$. Choosing left-invariant dreibeine gives "$V + A$" currents which are vectors on the tangent space. In terms of the above explicit coordinates, $U^{-1}\partial_\mu U = -i\ \tau^j J^j_\mu$, $J^i_\mu = -{v_a}^i \partial_\mu \varphi^a$, where ${v_a}^j = \sqrt{1-\varphi^2}\, g_{aj} + \varepsilon^{ajb} \varphi^b$. Note that these currents are pure gauge, or curvature-free, such that $\varepsilon^{\mu\nu} \left(\partial_\mu J^i_\nu + \varepsilon^{ijk} J^j_\mu J^k_\nu \right) = 0$.

In [5] we canonically mapped this model onto a dual σ model (DSM, with variables Φ^j) using the tangent space [7] generating functional $F[\Phi, \varphi] = \int\ dx\ \Phi^j J^{1\ j}[\varphi]$. Although we originally constructed F in the hamiltonian framework by some indirect reasoning, its structure is directly evident within the lagrangean framework as follows. Treating J as independent variables in $\mathcal{L}_{CM} = \frac{1}{2} J^j_\mu J^{\mu\ j}$, we impose the pure gauge condition by adding a Lagrange multiplier term, $\mathcal{L}_\lambda = \Phi^j\ \varepsilon^{\mu\nu} (\partial_\mu J^j_\nu + \varepsilon^{jkl} J^k_\mu J^l_\nu)$. Then, we complete a square for the J's and eliminate them from the dynamics in favor of the DSM [8]. But to do this, we must first write $\mathcal{L}_\lambda = \partial_\mu (\Phi^j\ \varepsilon^{\mu\nu}\ J^j_\nu) - \varepsilon^{\mu\nu}\ J^j_\nu\ \partial_\mu \Phi^j\ + \varepsilon^{jkl}\ \varepsilon^{\mu\nu}\ \Phi^j J^k_\mu J^l_\nu$. The total divergence term, integrated over a world-sheet with a constant time boundary, gives precisely our generating functional relating the CM to the DSM.

The supersymmetric extension (SCM) of the CM is [9]: $\mathcal{L}_{SCM} = \frac{1}{2} g_{ab}\ \partial_\mu \varphi^a \partial^\mu \varphi^b + \frac{1}{2} i g_{ab} \overline{\psi}^a \not{D} \psi^b + \frac{1}{8} (g_{ab} \overline{\psi}^a \psi^b)^2$, where

general principles and the previous coordinate choice yield $D_\mu\psi^b = \partial_\mu\psi^b + \Gamma^b_{cd}\partial_\mu\varphi^c\psi^d$, $\Gamma^a_{bc} = v^{aj}\partial_{(b}v_{c)}{}^j = \varphi^a g_{bc}$, $\partial_{[a}v_{b]}{}^j = \varepsilon^{jkl}v_a{}^k v_b{}^l$, as well as $g^{-1\ ab} = \delta^{ab} - \varphi^a\varphi^b = v^{aj}v^{bj}$, $v^{aj} = \sqrt{1-\varphi^2}\ \delta^{aj} + \varepsilon^{ajb}\varphi^b$. Again, we rewrite $\mathcal{L}_{SCM}$ in terms of tangent space quantities: the left-invariant tangent-space spinor is defined (e.g. [6]) by $\chi^j = v^j_a\psi^a$, and transforms as $\delta\chi^j = \varepsilon^{jkl}\xi^k\chi^l$ under a full $V+A$ transformation. Thus, its contribution to the corresponding current should be that of an isorotation. The tangent space lagrangean is then [6]:

$$\mathcal{L}_{SCM} = \frac{1}{2}\Big(J^j_\mu J^{\mu\ j} + i\overline{\chi}^j \partial\!\!\!/\ \chi^j + i\varepsilon^{jkl}\overline{\chi}^j J\!\!\!/^k\chi^l + \frac{1}{4}(\bar{\chi}^j\chi^j)^2\Big). \tag{1}$$

This tangent space formulation is utilized below in bridging over to the Supersymmetric Dual σ Model (SDSM).

The supersymmetry transformations leaving the SCM action invariant are $\delta\varphi^a = \bar{\epsilon}\psi^a$, $\delta\psi^a = (\frac{1}{2}\Gamma^a_{bc}\overline{\psi}^b\psi^c - i\partial\!\!\!/\varphi^a)\epsilon$. In tangent space, these become $\delta J^j_\mu = -\bar{\epsilon}(\partial_\mu\chi^j + \varepsilon^{jkl}J^k_\mu\chi^l)$, $\delta\chi^j = iJ\!\!\!/^j\epsilon - \frac{1}{2}\varepsilon^{jkl}(\gamma_p\epsilon\ \bar{\chi}^k\gamma_p\chi^l + \gamma_\mu\epsilon\ \bar{\chi}^k\gamma^\mu\chi^l)$. Either form induces the conserved supercurrent [9]

$$S_\mu = -i\partial\!\!\!/\varphi^a\gamma_\mu g_{ab}\psi^b\ = iJ\!\!\!/^j\gamma_\mu\chi^j\ . \tag{2}$$

The curvature-free vector currents are again the bosonic pure gauges J. However, the conserved vector currents now consist of these bosonic terms augmented by spinor bilinears: $C^i_\mu = J^i_\mu + K^i_\mu$, with $J^i_\mu = -v_a{}^i\partial_\mu\varphi^a$, $K^i_\mu = \frac{i}{2}\varepsilon^{ijk}\overline{\chi}^j\gamma_\mu\chi^k$.

Our new tangent space generator for a canonical transformation relating φ and χ at any fixed time to Φ and X (the bosons and fermions of the dual theory) is

$$F[\Phi, X, \varphi, \chi] = \int dx\ \left(\Phi^j J^{1\ j}[\varphi] - \frac{i}{2}\ \overline{X}^j\gamma^1\chi^j\right) \tag{3}$$

$= \int dx\Big(\Phi^i(\sqrt{1-\varphi^2}\,\frac{\overleftrightarrow{\partial}}{\partial x}\varphi^i + \varepsilon^{ijk}\varphi^j\frac{\partial}{\partial x}\varphi^k) - \frac{i}{2}\overline{X}^j\gamma^1\chi^j\Big)$.

Classically, the canonical conjugate to χ ($\pi_\chi \equiv \delta\mathcal{L}_{SCM}/\delta\partial_0\chi = -i\chi^\dagger/2$) is obtained from F as $-\delta F/\delta\chi = -iX^\dagger\gamma_p/2$, where $\gamma_p = \gamma^0\gamma^1$. So under the canonical transformation

$$\chi^j = \gamma_p X^j. \tag{4}$$

Likewise, the momentum conjugate to X is $\delta F/\delta X = -i\chi^\dagger\gamma_p/2$, leading to $\pi_X \equiv -iX^\dagger/2$, which specifies part of the dual lagrangean. This chiral rotation of the fermions reflects the duality transition of their bosonic superpartners, whose gradients map to curls (in the weak field limit). The equal-time anticommutation relations for Majorana spinors in tangent space, $\{\chi^j(x), \chi^k(y)\} = \{X^j(x), X^k(y)\} = 2\ \delta^{jk}\delta(x-y)$, are preserved by the above transformation, so it is properly identified as canonical.

Preservation of the canonical structure for the bosons is less evident. The current algebra (i.e. the Poisson brackets for the J) is, indeed, preserved in going over from the CM to the DSM. However, preservation of the current algebra is necessary but *not* sufficient for identification of such models [10]. The classical conjugate momentum of Φ^i is

$$\Pi_i = \frac{\delta F}{\delta\Phi^i} = \left(\sqrt{1-\varphi^2}\ \delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}} - \varepsilon^{ijk}\varphi^k\right)\frac{\partial\varphi^j}{\partial x} = J^{1\ i}, \tag{5}$$

and the conjugate of φ^i is

$$\varpi_i = -\frac{\delta F}{\delta\varphi^i} = \left(\sqrt{1-\varphi^2}\ \delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}} + \varepsilon^{ijk}\varphi^k\right)\frac{\partial\Phi^j}{\partial x} + \left(\frac{2}{\sqrt{1-\varphi^2}}\left(\varphi^i\Phi^j - \Phi^i\varphi^j\right) - 2\varepsilon^{ijk}\Phi^k\right)\frac{\partial}{\partial x}\varphi^j. \tag{6}$$

These classical relations map the SCM as defined by (1) to a dual theory, the SDSM. The dual theory is a supersymmetric extension of the bosonic DSM, which contains *torsion* [8,5], and is defined by the following lagrangean.

$$\begin{aligned}\mathcal{L}_{SDSM} &= \frac{i}{2}\overline{X}^j\partial\!\!\!/\ X^j + \frac{1}{8}(\overline{X}^j X^j)^2 - \frac{1}{2}\Big(\varepsilon_{\mu\lambda}\partial^\lambda\Phi^a - \tfrac{i}{2}\varepsilon^{ajk}\overline{X}^j\gamma_\mu X^k\Big)N^{\mu\nu}_{ab}\Big(\varepsilon_{\nu\rho}\partial^\rho\Phi^b - \tfrac{i}{2}\varepsilon^{blm}\overline{X}^l\gamma_\nu X^m\Big)\\ &= \frac{i}{2}\overline{X}^j\partial\!\!\!/\ X^j + \frac{i}{2}\varepsilon^{ijk}\overline{X}^i\mathcal{J}\!\!\!/^j X^k + \frac{3}{8}(\overline{X}^i X^i)\left((\overline{X}^j X^j) + \frac{(\varepsilon^{jkl}\overline{X}^j\gamma_p X^k\Phi^l + 2\overline{X}^j X^k\Phi^j\Phi^k)}{3\ (1+4\Phi^2)}\right)\\ &\quad + \frac{1}{2}\left(G_{jk}\ \partial_\mu\Phi^j\partial^\mu\Phi^k + E_{jk}\varepsilon^{\mu\nu}\partial_\mu\Phi^j\partial_\nu\Phi^k\right),\end{aligned} \tag{7}$$

where $N^{\mu\nu}_{ab} = g^{\mu\nu} G_{ab} + \varepsilon^{\mu\nu}\, E_{ab}$, $G_{ab} = (\delta^{ab} + 4\Phi^a\Phi^b)/(1+4\Phi^2)$, $E_{ab} = -2\varepsilon_{abc}\Phi^c/(1+4\Phi^2)$, $G^{-1\ ab} = (1+4\Phi^2)\,\delta^{ab} - 4\Phi^a\Phi^b = V^{aj}V^{bj}$, $V^{aj} = \delta^{aj} - 2\varepsilon^{ajb}\Phi^b$, and $V_a{}^j = G_{aj} + E_{aj}$. Since $\Phi_a = G_{ab}\Phi^b = \Phi^a$ and $\Phi^h = \Phi^a V_a{}^h$, it is unnecessary to distinguish the base- and tangent-space indices for the Φs.

The canonical momenta dictated by the lagrangeans $\mathcal{L}_{SCM}$ and $\mathcal{L}_{SDSM}$ are

$$\Pi_j = -N^{1\nu}_{jk}\Big(\varepsilon_{\nu\rho}\partial^\rho\Phi^k - \tfrac{i}{2}\varepsilon^{klm}\overline{X}^l\gamma_\nu X^m\Big), \tag{8}$$

$$\varpi_i = \left(\delta^{ij} + \frac{\varphi^i\varphi^j}{1-\varphi^2}\right)\frac{\partial\varphi^j}{\partial t} + (\sqrt{1-\varphi^2}\,\delta^{ij} + \frac{\varphi^i\varphi^j}{\sqrt{1-\varphi^2}} + \varepsilon^{ijl}\varphi^l)\ K^j_0[\chi]. \tag{9}$$

We could now replace Π and ϖ with (5,6) to get covariant expressions for the dual transformations of fields. Following that route leads to manifestly nonlocal expressions for φ and χ in terms of Φ and X. Instead, we focus on the identification of the curvature-free currents in the two theories, consistently with the above, an identification which turns out to be completely local.

The crucial transition bridge to the SDSM relies on the bosonic current

$$\mathcal{J}^\mu_j = -N^{\mu\nu}_{jk}\varepsilon_{\nu\lambda}\partial^\lambda\Phi^k = \frac{-1}{1+4\Phi^2}\Big((\delta^{jl} + 4\Phi^j\Phi^l)\varepsilon^{\mu\nu}\partial_\nu\Phi^l + 2\varepsilon^{jlk}\Phi^l\partial^\mu\Phi^k\Big), \tag{10}$$

which conjoins with the fermionic bilinear component into

$$J^{\mu\ j} = -N^{\mu\nu}_{jk}(\varepsilon_{\nu\lambda}\partial^\lambda\Phi^b - \tfrac{i}{2}\varepsilon^{klm}\overline{X}^l\gamma_\nu X^m) = \mathcal{J}^{\mu j} + N^{\mu\nu}_{jk}K^k_\nu. \tag{11}$$

(Note from (4), $K^j_\mu[\chi] = K^j_\mu[X]$.) This pivotal, covariant, classical relation linking J to Φ and X is derived in the canonical framework as follows.

The generating functional already automatically yielded the spacelike component $J^1 = \Pi = \mathcal{J}^1 + (N\cdot K)^1$. As in the bosonic model [5], use of the variation (6) yields a match for the timelike components as well, by virtue of

$$-\sqrt{1-\varphi^2}\,\varpi_i - \varepsilon^{ijk}\varphi^j\varpi_k = -\frac{\partial\Phi^i}{\partial x} - 2\varepsilon^{ijk}\Phi^j\Pi_k, \tag{12}$$

since both sides are equal to the mixed expression $-\partial\Phi^i/\partial x + 2\Phi^j\left(\varphi^j\partial\varphi^i/\partial x - \varphi^i\partial\varphi^j/\partial x\right)$ $+2\varepsilon^{ijk}\Phi^j\left(\varphi^k\partial(\sqrt{1-\varphi^2})/\partial x - \sqrt{1-\varphi^2}\partial\varphi^k/\partial x\right)$. Separation of bosonic from fermionic current pieces in Eq. (12) yields $J^i_0 - K^i_0 = \mathcal{J}^i_0 - 2\varepsilon^{ijk}\Phi^j(N\cdot K)^{k1}$. As a direct consequence,

$$J^0 = \mathcal{J}^0 + (N\cdot K)^0, \tag{13}$$

so that the covariant identification of currents (11) holds for the classical theory. Duality contrives to link the SCM and SDSM manifolds nontrivially. *Bosons in one theory are composites of both bosons and fermions in the dual theory—a fermion-boson symphysis.* Such mixings of boson and fermion bilinear components of currents are not unfamiliar in standard canonical bosonization schemes [1].

Predicated on the above identification of the vector currents, the supercurrent (2) for SDSM now simply reads

$$\mathfrak{S}_\mu = i\gamma_p(\not{\mathcal{J}}^j + \not{N}\cdot K^j)\gamma_\mu X^j. \tag{14}$$

Consequently, the respective supercharges identify in the two models, and whence their squares, viz. the hamiltonians. Likewise, by supertransforming the supercurrents, the respective energy-momentum tensors identify, including the respective hamiltonian densities. Hence, the SDSM is dynamically equivalent to the SCM in phase space. (Care must be taken in the identifications of covariant quantities, however, such as lagrangean densities. Such identifications are direct only in terms of canonical variables, and thus may only be valid “on-shell”, since supplanting πs with $\partial/\partial t$s requires Hamilton’s equations.)

Naturally, the bosonic limits ($\chi = X = 0$) of the above actions are the models already connected in [5]. However, the fermionic limits ($\varphi = \Phi = 0$) are both the same Gross-Neveu model, but with different normalizations of the interaction term, $1/8 \longrightarrow 3/8$, respectively, as a consequence of the banished bosons in the above symphysis. This change of the fermion coupling under such a singular transformation evokes the change of coupling (i.e. inversion of radii) under duality in bosonic models (on tori).

F is "covariant" with respect to supersymmetry transformations, in that supertransforming its SCM variables is the same as supertransforming its SDSM ones, essentially by construction of $\mathfrak{S}$. Furthermore, the full classical Poisson Bracket algebra of all generators (F, H, Q, etc.) closes, but we will not discuss the details here. We are more interested in the quantum algebra of the generators, to which we now turn.

The generator F actually relates the SCM to the SDSM not only at the classical level, but also at the quantum level as well, just as in the case of Liouville theory [4]. In general, a classical generating functional is the first step in Dirac's implementation [11] of the corresponding canonical transformation for the quantum theory. The present situation appears to be special, as was the purely bosonic case [5]. We exponentiate the classical generating functional to obtain a candidate for the transformation functional connecting wave functionals for the SCM to those for the SDSM:

$$T[\Phi, X, \varphi, \chi] \equiv e^{iF[\Phi,X,\varphi,\chi]}. \tag{15}$$

In a basis where the fields are diagonalized, this T is supposed to link states in the two theories by

$$\langle \Phi, X|t\rangle = \int T[\Phi, X, \varphi, \chi]\ \langle \varphi, \chi|t\rangle\ d\varphi\ d\chi\ . \tag{16}$$

The integrals on the RHS are over all field configurations at a fixed time, t. It is in this sense that the two theories are canonically equivalent at the quantum level [12].

We now argue that our candidate $T = \exp(iF)$ provides exactly this link by considering the action of local currents on the wave functionals. It suffices to consider the vector currents and the supercurrents. We show the action of the SDSM currents on the SDSM wave functionals reduces to the action of the corresponding SCM currents on the SCM wave functionals when $T = \exp(iF)$ is used in (16).

This follows through the use of integration by parts under the functional integrals in (16), and the fact that T undergoes the same change when acted on by either set of currents. We need only consider how the currents act on T, as explained for the bosonic components of the currents in [5]. There, as here, we express the classical currents in terms of fields and their classical conjugate momenta, and then replace the classical conjugate momenta with functional derivatives: $\varpi \to -i\delta/\delta\varphi$, $\Pi \to -i\delta/\delta\Phi$. A similar procedure applies to the spinors as well, with some care to maintain their Majorana character. Since $-i\ \chi^\dagger/2$ is the classical conjugate of χ, we would replace $\chi^\dagger \to 2\delta/\delta\chi$, given the usual $\{\pi_\chi(x), \chi(y)\} = -i\delta(x-y)$. But in the Majorana representation, $\chi^\dagger = \chi^T$. Thus, a prescription consistent with both is to replace $\chi \to (\chi/\sqrt{2}\ + \sqrt{2}\ \delta/\delta\chi^\dagger)$ wherever the spinors appear in the currents. These prescriptions yield currents as functional differential operators acting on the space of wave functionals. It then follows that

$$\mathfrak{S}_\mu \langle \Phi, X|t\rangle = \int \ \left(\mathfrak{S}_\mu e^{iF[\Phi,X,\varphi,\chi]}\right)\ \langle \varphi, \chi|t\rangle\ d\varphi\ d\chi\ = \int\ e^{iF[\Phi,X,\varphi,\chi]}\ S_\mu \langle \varphi, \chi|t\rangle\ d\varphi\ d\chi\ , \tag{17}$$

upon integrating by parts and discarding any surface terms in the functional integration. The steps in the calculation are the same as in the purely bosonic case [5] with the additional identities $(X^k + 2\ \delta/\delta X^{k\ \dagger})\exp(\int dx\ \overline{X}^j\gamma^1\chi^j/2) = (X^k + \gamma_p\chi^k)\exp(\int dx\ \overline{X}^j\gamma^1\chi^j/2) = \gamma_p\ (\chi^k - 2\ \delta/\delta\chi^{k\ \dagger})\exp(-\int dx\ \overline{\chi}^j\gamma^1 X^j/2)$. Finally, note $\delta/\delta\chi^\dagger$ in the last term changes sign upon integration by parts. Results similar to (17) hold for all the other local currents in the two models. Thus, we conclude that our candidate transformation functional induces the desired interchange of SDSM and SCM local currents acting on the corresponding wave functionals.

Our argument for the quantum case is formal, obviously, since we have ignored the UV divergences that inevitably arise in such manipulations. To treat the problem with more care, we would need to discuss the renormalization of T, F, and also the wave functionals for both the SCM and the SDSM, in continuing parallel with the Liouville case. This is beyond the scope of the present paper. However, we may conjecture that the renormalization proceeds for T in terms of the SCM manifold geometry by nothing but the usual renormalization of the dreibein that appears in F, and perhaps by rescaling (or at worst more complicated field redefinitions of) the tangent space fields. In general, we would also expect the normalizations of the wave functionals to be changed by the transformation but only through energy-dependent factors. If this is indeed so, the formal arguments above have given the correct structure of the exact answer, just as for the Liouville theory.

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